It was not until November, 1899, that opportunity was found to look over the letter to M. Hermite (Comptes Rendus 127, 1898, page 539). On examination difficulties were found in it, which the writer in a letter, November 9, 1899, stated to Professor Gordan, together with a sketch of proposals for their removal. The privilege of publishing the letter of October 17, 1898, was also asked. In reply November 21, 1899, Professor Gordan wrote as follows: "Ihre Behandlung der Aufgabe ist richtig und stimmt im Wesentlichen mit derjenigen ueberein, welche ich in Muenchen vorgetragen habe. Meine Arbeit habe ich vor ca 6 Monaten an die Annalen gesandt, sie ist aber noch nicht gedruckt; sobald diess geschieht werde ich Ihnen ein Exemplar senden. Wenn sie selbst etwas darueber schreiben wollen, so wird mich das sehr freuen. Mine Darstellung in dem Briefe an Hermite war noch unreif; ich habe sie seither verbessert."

The writer has not yet received a copy of Professor Gordan's Munich paper, and what follows is substantially the elaboration of the sketch sent to Professor Gordan November 9, 1899. The working out of his results is given in much more detail than was needed in the sketch.

§ 2. THE FUNCTIONS f, ϕ, ψ.

The functions

$$f=x^m+a_1x^{m-1}+\ldots,+a_m\ldots\ldots(1)$$

$$\phi=x^n+b_1x^{n-1}+\ldots.+b_n\ldots\ldots\quad(2)$$

$$\psi=1+b_1x+\ldots.+b_mx^n\ldots\ldots(3)$$

are three integral functions of the mth and nth degrees, respectively. Let the roots of f be α_1, α_2, $\ldots.\alpha_m$, and those of ϕ, β_1, β_2, $\ldots.\beta_n$, then the roots of ψ are $1/\beta_1$, $1/\beta_2$, $\ldots.1/\beta_n$, *i. e.*, the roots of ψ are the reciprocals of roots of ϕ. We have:

$$\psi=(1-\beta_1x)(1-\beta_2x)\ldots.(1-\beta_nx)\ldots\ldots\quad(4).$$

§ 3. THE EXPONENTIAL FORM OF ψ.

By developing $\log\psi=\sum_{i=1}^{i=n}\log(1-\beta_ix)\ldots\ldots(5)$ which we may always certainly do, so long as we take x to satisfy the inequality $\mathrm{mod}(\beta x)<1$, where β denotes that root for which $\mathrm{mod}(\beta)\geq\mathrm{mod}(\beta_i)$, we get

$$\log\psi=-\sum_{r=1}^{r=\infty}(1/r)s_r(b)x^r\ldots\ldots(6)$$

where $s_r(b)$ denotes the sum of the rth powers of the β_i's, or we may write

$$\psi(x)=e^{-\sum_{r=1}^{r=\infty}(1/r)s_r(b)x^r}\ldots\ldots(7)$$

for values of x satisfying the inequality $\mathrm{mod}(x)<\mathrm{mod}(1/\beta)$, and this is the exponential form of ψ.

§ 4. THE RESULTANT OF f AND ψ.

If we choose those functions f, whose roots satisfy the inequality $\text{mod}(\alpha) \leqq \text{mod}(x) < \text{mod}(1/\beta)$, where α is that root for which $\text{mod}(\alpha) \geqq \text{mod}(\alpha_i)$, then $\psi(\alpha_i) = e^{-\sum_{r=1}^{r=\infty} (1/r) s_r(b) \alpha_i^r}$, and the resultant of f and ψ, $\prod_{i=1}^{i=m} \psi(\alpha_i)$, becomes

$$R = e^{-\sum_{r=1}^{r=\infty} (1/r) s_r(a) s_r(b)} \quad \ldots\ldots (8),$$

where $s_r(a)$ denotes the sum of the rth powers of the α_i's. The formula (8) is *the exponential form* of the resultant of f and ψ, and it has been shown that it is true for such functions f and ψ which are so related that the inequality $\text{mod}(\alpha) < \text{mod}(1/\beta)$ is satisfied.

§ 5. THE EXPANSION OF R.

If we expand the exponential form given in the last section we obtain :

$$R = \sum_{\nu=0}^{\nu=\infty} \frac{(-1)^\nu}{\nu!} \left(\sum_{\lambda=1}^{\lambda=\infty} (1/\lambda) s_\lambda(a) s_\lambda(b) \right)^\nu \ldots\ldots (9).$$

By the multinomial theorem,

$$\left(\sum_{\lambda=1}^{\lambda=\infty} (1/\lambda) s_\lambda(a) s_\lambda(b) \right)^\nu = \Sigma \frac{\nu!}{\lambda_1^{\rho_1} \lambda_2^{\rho_2} \ldots . \rho_1! \rho_2! \ldots .} s_{\lambda_1}^{\rho_1}(a) s_{\lambda_1}^{\rho_1}(b) s_{\lambda_2}^{\rho_2}(a) s_{\lambda_2}^{\rho_2}(b) \ldots .$$

where $\rho_1 + \rho_2 + \ldots . = \nu$, hence,

$$R = \sum_{\nu=0}^{\nu=\infty} \Sigma \frac{(-1)^{\rho_1+\rho_2+\ldots\ldots}}{\lambda_1^{\rho_1} \lambda_2^{\rho_2} \ldots . \rho_1! \rho_2! \ldots .} (s_{\lambda_1}(a) s_{\lambda_1}(b))^{\rho_1} (s_{\lambda_2}(a) s_{\lambda_2}(b))^{\rho_2} \ldots\ldots (10).$$

We shall call this *the transcendental form* of the resultant. And we here add this remark : Though the previous developments upon which (10) depends have not been shown to be true for all functions f and ψ, yet when the left member R and the symmetric functions $s_r(a)$, $s_r(b)$ are replaced, for admissible functions, by their values in terms of the a's and b's, the relation (10) is seen to be an identity in terms of the latter, and therefore true for all functions f and ψ.

§ 6. COMPLETE GENERALITY OF THE TRANSCENDENTAL FORM.

From the foregoing sections it will be observed that the transcendental form of the resultant is precisely the same in terms of the s's for every binary resultant whatever. In obtaining the transcendental expression for R, we stated it is true with forms f and ψ of the mth and nth degrees, but we should have come precisely to the same transcendental form in terms of the s's, if we had used functions of any other degrees. This is the reason why we omitted to affect the symbol R with the subscripts m and n. *The transcendental form expresses*

all binary resultants formally in the same form, and is a universal formula comprehending them all. It is the most general formula for binary resultants.

§ 7. THE FINITE FORM OF THE RESULTANT.

As the resultant $R_{m,n}$ of two forms of the mth and nth degrees of the form f and ψ, must contain terms of weight mn in the a's, but no terms of higher weight, it follows that the terms of the transcendental form of weight greater than mn, taken together are, by the remark of § 5, identically equal to zero, when the s's are replaced by their values in terms of the a's and b's, though this will not appear explicitly or formally in terms of the s's,* and for the finite form we have,

$$R_{m,n}=\Sigma\Sigma\frac{(-1)^{\rho_1+\rho_2+\ldots}}{\lambda_1^{\rho_1}\lambda_2^{\rho_2}\ldots\rho_1!\rho_2!\ldots}(s_{\lambda_1}(a)s_{\lambda_1}(b))^{\rho_1}(s_{\lambda_2}(a)s_{\lambda_2}(b))^{\rho_2}\ldots\ldots(11)$$

where the summation is to be extended over all values of the λ's and ρ's for which $\lambda_1\rho_1+\lambda_2\rho_2+\ldots\leqq mn$.

Example: Let
$$f=x^2+a_1x+a_2$$
$$\phi=x+b_1$$
$$\psi=1+b_1x.$$

By the preceding formula or directly by elementary methods we find

$$R_{f,\,\psi}=R_{2,\,1}=1-s_1(a)s_1(b)+\tfrac{1}{2}(s_1(a)s_1(b))^2-\tfrac{1}{2}s_2(a)s_2)b).$$

The resultant of any two binary forms may be expressed in this manner by (11), for it is only necessary to supply for the $s(b)$'s those s's which belong to the form whose roots are the reciprocals of those of the given second form.

§ 8. GENERALITY OF THE FORMULA (11).

The form (11) which we may write, by using Gordan's abbreviations,

$$R_{m,\,n}=\Sigma\frac{(-1)^{\rho_1+\rho_2+\ldots}}{\lambda_1^{\rho_1}\lambda_2^{\rho_2}\ldots\rho_1!\rho_2!\ldots}S_\lambda(a)\,S_\lambda(b)\ \ldots\ldots(12)$$

with $\lambda_1\rho_1+\lambda_2\rho_2+\ldots\leqq mn$, represents formally all resultants $R_{\mu,\nu}$, for which $\mu\nu=mn$, in terms of the s's in precisely the same form; therefore it is still a very general and elastic form. In order to make it definite, one substitutes for the s's their values in terms of the a's and b's for definite equations whose degrees m_1, n_1, satisfy the condition $m_1n_1=mn$. Then it represents a definite resultant R_{m_1,n_1} of the given kind.

Example: $R_{12,1}$, $R_{1,12}$, $R_{3,\,4}$, $R_{4,\,3}$, $R_{2,\,6}$, and $R_{6,\,2}$, all have formally the same expression in terms of the s's.

The expression for $R_{m,n}$ will represent something which is not at all a re-

*And as Professor White has suggested, it is also well to state, what is involved in this, that all terms of the same weight beyond those of weight mn, are seen to vanish identically by themselves, when the s's are replaced by a's and b's.

sultant (something more or less than a resultant) if values of the s's in terms of the a's and b's are substituted which belong to equations where $m_1 n_1$ not equal to mn. In consequence of this the symbol $R_{m,n}$ includes not only resultants but also other expressions not resultants. We use the symbol in the following section in the enlarged sense.

§ 9. PARTIAL DIFFERENTIAL EQUATION WITH RESPECT TO THE s'S.

If we differentiate the form

$$R_{m,n} = \Sigma \frac{(-1)^{\rho_1+\rho_2+\dots}}{\lambda_1^{\rho_1}\lambda_2^{\rho_2}\dots\rho_1!\,\rho_2!\dots}(s_{\lambda_1}(a)s_{\lambda_1}(b))^{\rho_1}(s_{\lambda_2}(a)s_{\lambda_2}(b))^{\rho_2}\dots\dots$$

of (11) partially with respect to $s_{\lambda_r}(a)$, we obtain

$$\lambda_r\frac{\partial R_{m,n}}{\partial s_{\lambda_r}(a)} = (-1)s_{\lambda_r}(b)\,\Sigma\frac{(-1)^{\rho_1+\rho_2+\dots+(\rho_r-1)+\dots}}{\lambda_1^{\rho_1}\lambda_2^{\rho_2}\dots\lambda_r^{\rho_r-1}\dots\rho_1!\,\rho_2!\dots(\rho_r-1)!\dots}$$

$$(s_{\lambda_1}(a)s_{\lambda_1}(b))^{\rho_1}\dots(s_{\lambda_r}(a)s_{\lambda_r}(b))^{\rho_r-1}\dots\dots \quad (13)$$

with $\lambda_1\rho_1+\lambda_2\rho_2+\dots\leq mn-\lambda_r$, and hence our equation is

$$\lambda_r\frac{\partial R_{m,n}}{\partial s_{\lambda_r}(a)} = (-1)s_{\lambda_r}(b)R_{m',n'}, \text{ with } m'n' = mn-\lambda_r.$$

By repeating this process until we have differentiated ρ_r times we get

$$\frac{\lambda_r^{\rho_r}\partial^{\rho_r}R_{m,n}}{\partial s_{\lambda_r}^{\rho_r}(a)} = (-1)^{\rho_r}s_{\lambda_r}^{\rho_r}(b)R_{m',n'}\dots\dots(14)$$

where m', $n' = mn-\lambda_r\rho_r$, and in a similar way we obtain more generally still,

$$(-1)^{\rho_1+\rho_2+\dots}\lambda_1^{\rho_1}\lambda_2^{\rho_2}\dots\frac{\partial^{\rho_1+\rho_2+\dots}R_{m,n}}{\partial s_{\lambda_1}^{\rho_1}\partial s_{\lambda_2}^{\rho_2}\dots\dots} = s_{\lambda_1}^{\rho_1}s_{\lambda_2}^{\rho_2}\dots R_{m',n'}\dots\dots \quad (15).$$

$$m'n' = mn-(\lambda_1\rho_1+\lambda_2\rho_2+\dots)\dots\dots \quad (16)$$

where the differentiation may be distributed between the $s_\lambda(a)$'s and the $s_\lambda(b)$'s, and $R_{m',n'}$ may under certain circumstances be a resultant.

Examples: 1. $\dfrac{\partial R_{3,2}}{\partial s_2(a)} = -\frac{1}{2}s_2(b)R_{r,r'}$, where $rr'=4$, but $R_{r,r'}$ is neither the resultant $R_{4,1}$, nor $R_{2,2}$; the presence of terms $s_\lambda(a)$ where $\lambda>1$, which must contain b_2 prevents the former; the presence of terms $s_\lambda(a)$ where $\lambda>2$, which must contain a_3 prevents the latter,

2. $\dfrac{\partial R_{6,1}}{\partial s_5(a)} = -\frac{1}{5}s_5(b)R_{r,r'}$, where $rr'=1$, and here $R_{r,r'}=R_{1,1}$, is the resultant of $x+a_1$, and $1+b_1x$.

§ 10. DIFFERENTIATION OF THE TRANSCENDENTAL FORM.

The same differentiation applied to the transcendental form (10), or what is the same thing (as is seen by performing the differentiation in both cases) to the exponential form (8) is questionable. By (8) we have $R=e^{-\Sigma(1/r)s_r(a)s_r(b)}$. Differentiating with respect to $s_\lambda(a)$,

$$\frac{\lambda\partial R}{\partial s_\lambda(a)}=(-1)e^{-\Sigma(1/r)s_r(a)s_r(b)}s_\lambda(b), \text{ or } \lambda\frac{\partial R}{\partial s_\lambda(a)}=(-1)s_\lambda(b)R \quad \ldots\ldots(17),$$

with a corresponding formula

$$(-1)^{\rho_1+\rho_2+\ldots}\lambda_1{}^{\rho_1}\lambda_2{}^{\rho_2}\ldots\frac{\partial^{\rho_1+\rho_2+\ldots}R}{\partial s_{\lambda_1}{}^{\rho_1}\partial s_{\lambda_2}{}^{\rho_1}\ldots}=s_{\lambda_1}{}^{\rho_1}s_{\lambda_2}{}^{\rho_2}\ldots R\ldots\ldots(18)$$

for the repitition. If (18) is allowed to stand it must be with the understanding that it is the universal type of all differential equations of the kind, that the R involved is the transcendental form, and that in any particular case such values in finite form must be picked out from the infinity of values which R admits, as will throw the indefinite form of (18) into that of (15) by which (18), if at all allowed, must be interpreted and explained.

II. APPLICATION TO SYMMETRIC FUNCTIONS.

§ 11. THE GENERAL TERM IN THE EXPANSION OF $\Sigma\alpha_1{}^{\kappa_1}\alpha_2{}^{\kappa_2}\ldots\alpha_\nu{}^{\kappa_\nu}$.

1. *In Terms of the s's.*

In terms of symmetric functions, the resultant $R_{m,n}$ of f and ψ consists of terms of the form $b_{\kappa_1}b_{\kappa_2}\ldots b_{\kappa_\nu}\Sigma\alpha_1{}^{\kappa_1}\alpha_2{}^{\kappa_2}\ldots\alpha_\nu{}^{\kappa_\nu}$. On the other hand we may express $R_{m,n}$ by (11) as a sum of terms of the form

$$\frac{(-1)^{\rho_1+\rho_2\ldots\rho_\kappa}}{\lambda_1\lambda_2\ldots\lambda_\mu\rho_1!\rho_2!\ldots\rho_\kappa!}s_{\lambda_1}(a)s_{\lambda_2}(a)\ldots s_{\lambda_\mu}(a)s_{\lambda_1}(b)s_{\lambda_2}(b)\ldots s_{\lambda_\mu}(b)\ldots\ldots(19)$$

by slightly changing the notation, ρ_1 being the number of λ's equal to λ_1, ρ_2 the number equal to λ_2, etc, and the summation to extend over all values of the λ's for which $\lambda_1+\lambda_2+\ldots\lambda_\mu\leq mn$. In the term of $R_{m,n}$, as so expressed, we pick out the portion containing $b_{\kappa_1}b_{\kappa_2}\ldots b_{\kappa_\nu}$ from the product $s_{\lambda_1}(b)s_{\lambda_2}(b)\ldots s_{\lambda_\mu}(b)$, and obtain

$$(-1)^{\nu+\rho_1-\rho_2+\ldots\ldots}C_{\lambda,\kappa}b_{\kappa_1}b_{\kappa_2}\ldots b_{\kappa_\mu}s_{\lambda_1}(a)s_{\lambda_2}(a)\ldots s_{\lambda_\mu}(a)$$

where $\kappa_1+\kappa_2+\ldots\kappa_\nu=\lambda_1+\lambda_2+\ldots\lambda_\mu$. By Waring's formula,

$$s_{\lambda_r}(b)=\lambda_r\Sigma(-1)^{\tau_1+\tau_2+\ldots\tau_n}\frac{(\tau_1+\tau_2+\ldots\tau_n-1)!}{\tau_1!\,\tau_2!\ldots\tau_n!}b_1{}^{\tau_1}\,b_2{}^{\tau_2}\ldots\,b_n{}^{\tau_n}$$

with $\tau_1+2\tau_2+\ldots n\tau_n=\lambda_r$. Among the terms of $s_{\lambda_r}(b)$ is

$$\frac{\lambda_r(-1)^{\nu_r}(\nu_r-1)!}{\tau_1!\tau_2!\ldots}b_{\kappa_{r1}}\,b_{\kappa_{r2}}\ldots b_{\kappa_{r\nu}}$$

where κ_{r1}, $\kappa_{r2}\ldots$ form a set of numbers found among κ_1, κ_2, $\ldots\kappa_\nu$, and τ_1

of the set are equal to $\varkappa_{r1}$, τ_2 to $\varkappa_{r2}$, etc., and where $\varkappa_{r1}+\varkappa_{r2}+\ldots.\varkappa_{r_{\nu_r}}=\lambda_r$. In the same way from from $r=1$, to $r=\mu$, we obtain from μ sets with $\gamma_1+\gamma_2+\ldots.$ $\gamma_\mu=\nu$, similar factors, and therefore a term containing $b_{\kappa_1}b_{\kappa_2}\ldots.b_{\kappa_\nu}$ in (19) is

$$\frac{(-1)^{\rho_1+\rho_2+\ldots.\ \nu}}{\rho_1!\,\rho_2!\ldots.\tau_1!\,\tau_2!\ldots.}(\nu_1-1!)(\nu_2-1)!\ldots(\nu_\mu-1)!s_{\lambda_1}(a)..s_{\lambda_\mu}(a)b_{\kappa_1}b_{\kappa_2}..b_{\kappa_\nu}..(20)$$

In like manner other decompositions, if there be such, of the $\nu\varkappa$'s into μ sets, where the sum of the sets is equal to λ_1, $\lambda_2\ldots.\lambda_\mu$ respectively, will give similar expressions, and the entire coefficient of $b_{\kappa_1}b_{\kappa_2}..b_{\kappa_\nu}s_{\lambda_1}(a)..s_{\lambda_\mu}(a)$ will be

$$\Sigma\frac{(-1)^{\rho_1+\rho_2+\ldots.\ \rho_\kappa+\nu}}{\rho_1!\rho_2!\ldots\rho_\kappa!\,\tau_1!\tau_2!\tau_r!\ldots}(\nu_1-1)!(\nu_2-1)!\ldots(\nu_\mu-1)!=(-1)^{\rho_1+\rho_2+\ldots.^\nu}C_{\lambda,\kappa}\ldots(21).$$

Another term $s_{\mu_1}(a)s_{\mu_2}(a)\ldots.s_{\mu_\kappa}(a)b_{\kappa_1}b_{\kappa_2}\ldots.b_{\kappa_\nu}$ will have a coefficient similarly formed, etc. Equating the terms in both developments of $R_{m,n}$ which contain $b_{\kappa_1}b_{\kappa_2}\ldots.b_{\kappa_\nu}$ and dividing out this product, we have,

$$\Sigma\alpha_1{}^{\kappa_1}\alpha_2{}^{\kappa_2}\ldots.\alpha_\nu{}^{\kappa_\nu}=\Sigma(-1)^{\nu+\rho_1+\rho_2+\ldots.\rho_\kappa}C_{\lambda,\kappa}S_\lambda(a)\ldots.(22).$$

It will be seen that this formula agrees with that given in Faà di Bruno's Binäre Formen, page 8, except that the terms are grouped differently in the summation, formula (22) grouping the terms with respect to the numerical values of the subscripts of the s's, the latter grouping them with respect to the number of elements entering into a combination to form the subscript. The coefficient $(-1)^{\nu+\rho_1+\rho_2+\ldots.}C_{\lambda,\kappa}$ of (21) is the coefficient of the general term $S_\lambda(a)$ in the expansion of $\Sigma\alpha_1{}^{\kappa_1}\alpha_2{}^{\kappa_2}\ldots.\alpha_\nu{}^{\kappa_\nu}$ in terms of the s's.

2. *In Terms of the a's.*

By using the preceding formula (22), we easily obtain a formula for the expansion of $\Sigma\alpha_1{}^{\kappa_1}\alpha_2{}^{\kappa_2}\ldots.\alpha_\nu{}^{\kappa_\nu}$ in terms of the a's. We have

$$s_{\lambda_r}(a)=\lambda_r\Sigma(-1)^{\tau_1+\tau_2+\ldots.\tau_m}\frac{(\tau_1+\tau_2+\ldots.\tau_m-1)!}{\tau_1!\,\tau_2!\ldots.\tau_m!}a_1{}^{\tau_1}\,a_2{}^{\tau_2}\ldots.a_m{}^{\tau_m}$$

with $\tau_1+2\tau_2+\ldots.m\tau_m=\lambda_r$. Among the terms of $s_{\lambda_r}(a)$ is

$$\frac{(-1)^{\sigma_r}\lambda_r(\sigma_r-1)!}{\varkappa_1!\varkappa_2!\ldots.\varkappa_\rho!}a_{\mu_{r_1}}a_{\mu_{r_2}}\ldots.a_{\mu_{r\sigma_r}}$$

where $\mu_{r1}+\mu_{r2}+\ldots\mu_{r\sigma_r}=\lambda_r$, the $\varkappa$'s are for repetitions in the a's, $\sigma_1+\sigma_2+\ldots$ $\sigma_\mu=\sigma$, and μ_{r1}, $\mu_{r2}\ldots.$ are found among μ_1, μ_2, $\ldots.\mu_\sigma$; similarly for other factors. Then a term in $(-1)^{\nu+\rho_1+\rho_2+\ldots.}C_{\lambda,\kappa}S_\lambda(a)$ containing $a_{\mu_1}a_{\mu_2}\ldots.a_{\mu_\sigma}$ is $(-1)^{\sigma+\nu+\rho_1+\rho_2+\ldots.}$

$$\frac{\lambda_1\lambda_2\ldots.}{\varkappa_1!\varkappa_2!\ldots.}C_{\lambda,\kappa}(\sigma_1-1)!(\sigma_2-1)!\ldots.(\sigma_\mu-1)!a_{\mu_1}a_{\mu_2}\ldots.a_{\mu_\sigma}\ldots.(23).$$

The sum of all numerical coefficients of the form contained in (23) is the complete coefficient of $a_{\mu_1}a_{\mu_2}\ldots.a_{\mu_\sigma}$ in the expansion of $\Sigma\alpha_1{}^{\kappa_1}\alpha_2{}^{\kappa_2}\ldots.\alpha_\nu{}^{\kappa_\nu}$ and this in his notation Gordan expresses as

$$\Sigma(-1)^{\sigma+\nu+\rho_1+\rho_2+\ldots.}\frac{g_\lambda}{\tau_1!\,\tau_2!\ldots.}(\varkappa_{11}\varkappa_{12}\ldots.\mid\varkappa_{21}\varkappa_{22}\ldots.\mid\ldots.)$$

$$(\mu_{11}\mu_{12}\ldots.\mid\mu_{21}\mu_{22}\ldots.\mid\ldots.)\ldots.(24).$$

By means of this formula the general coefficient in the expansion of the binary resultant in terms of the a's and b's can of course be independently expressed.

Elmira, New York, December 19, 1899.

INTEGRATION OF ELLIPTIC INTEGRALS.

By G. B. M. ZERR. A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

[Continued from December Number.]

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^4\theta\cos^4\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} = \int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta\cos^4\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} - \int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta\cos^6\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}.$$

$\therefore$ From (12) and (19) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta\cos^6\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} = \frac{1}{105e^8}[(48-128e^2+103e^4-15e^6)E(e, \tfrac{1}{2}\pi)$$

$$-4(12-38e^2+41e^4-15e^6)F(e, \tfrac{1}{2}\pi)]\ldots\ldots(20).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta\cos^6\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} = \int_0^{\frac{1}{2}\pi} \frac{\cos^6\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} - \int_0^{\frac{1}{2}\pi} \frac{\cos^8\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}.$$

$\therefore$ From (13) and (20) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \frac{\cos^8\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} = \frac{1}{105e^8}[(48-208e^2+353e^4-298e^6+105e^8)F(e, \tfrac{1}{2}\pi)$$

$$-2(24-92e^2+132e^4-88e^6)E(e, \tfrac{1}{2}\pi)] \ldots. (21).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\sin^6\theta d\theta = \int_0^{\frac{1}{2}\pi} \frac{\sin^6\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} - e^2\int_0^{\frac{1}{2}\pi} \frac{\sin^8\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}.$$

$\therefore$ From (10) and (17) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\sin^6\theta d\theta = \frac{1}{105e^6}[(8+5e^2+11e^4-24e^6)F(e, \tfrac{1}{2}\pi)$$

$$-(8+9e^6+16e^4-48e^6)E(e, \tfrac{1}{2}\pi)]\ldots\ldots (22).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\sin^4\theta\cos^2\theta d\theta = \int_0^{\frac{1}{2}\pi} \frac{\sin^4\theta\cos^2\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} - e^2\int_0^{\frac{1}{2}\pi} \frac{\sin^6\theta\cos^2\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}$$

$\therefore$ From (11) and (18) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\sin^4\theta\cos^2\theta d\theta = \frac{1}{105e^6}[(8-5e^2-5e^4+8e^6)E(e, \tfrac{1}{2}\pi)$$

$$-(8-9e^2-3e^4+4e^6)F(e, \tfrac{1}{2}\pi)]\ldots\ldots (23).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\sin^2\theta\cos^4\theta d\theta = \int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta\cos^4\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} - e^2\int_0^{\frac{1}{2}\pi} \frac{\sin^4\theta\cos^4\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}$$

$\therefore$ From (12) and (19) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\sin^2\theta\cos^4\theta d\theta = \frac{1}{105e^6}[(3-23e^2+18e^4-3e^6)F(e, \tfrac{1}{2}\pi)$$

$$-(8-19e^2+9e^4-6e^6)E(e, \tfrac{1}{2}\pi)] \quad \ldots\ldots (24).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\cos^6\theta d\theta = \int_0^{\frac{1}{2}\pi} \frac{\cos^6\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} - e^2\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta\cos^6\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}.$$

$\therefore$ From (13) and (20) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\cos^6\theta d\theta = \frac{1}{105e^6}[(8-32e^2+58e^4+15e^6)E(e, \tfrac{1}{2}\pi)$$

$$-(8-32e^2+74e^4-45e^6)F(e, \tfrac{1}{2}\pi)] \ldots\ldots (25).$$

Let $n=3$ in (10_0), then $A_5 = \frac{8}{9}\cdot\frac{1+e^2}{e}\cdot A_4 - \frac{7}{9}A_3$.

$$\therefore A_5 = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos 5\varphi d\varphi}{[1+e^2-2e\cos\varphi]^{\frac{1}{2}}}$$

$$= \frac{4}{315\pi e^5}[(128+40e^2+39e^4+44e^6+64e^8)F(e, \tfrac{1}{2}\pi)$$

$$-(128+104e^2+99e^4+104e^6+128e^8)E(e, \tfrac{1}{2}\pi)] \ldots\ldots (19_0).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^{10}\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} = \frac{1}{315e^{10}}[(128+40e^2+39e^4+44e^6+64e^8)F(e, \tfrac{1}{2}\pi)$$

$$-(128+104e^2+99e^4+104e^6+128e^8)E(e, \tfrac{1}{2}\pi)] \ldots\ldots (26).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^8\theta\cos^2\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} = \frac{1}{315e^{10}}[(128-40e^2-21e^4-16e^6-16e^8)E(e, \tfrac{1}{2}\pi)$$

$$-(128-104e^2-9e^4-7e^6-8e^8)F(e, \tfrac{1}{2}\pi)] \ldots\ldots (27).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^6\theta\cos^4\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} = \frac{1}{315e^{10}}[(128-248e^2+111e^4+5e^6+4e^8)F(e, \tfrac{1}{2}\pi)$$

$$-(128-184e^2+27e^4+11e^6+8e^8)E(e, \tfrac{1}{2}\pi)] \ldots\ldots (28).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^4\theta\cos^6\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} = \frac{1}{315e^{10}}[(128-328e^2+243e^4-25e^6-10e^8)E(e, \tfrac{1}{2}\pi)$$

$$-(128-392e^2+399e^4-130e^6-5e^8)F(e, \tfrac{1}{2}\pi)] \ldots\ldots (29).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta\cos^8\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} = \frac{1}{315e^{10}}[(128-536e^2+855e^4-622e^6+175e^8)F(e, \tfrac{1}{2}\pi)$$

$$-(128-472e^2+627e^4-334e^6+35e^8)E(e, \tfrac{1}{2}\pi)] \ldots\ldots (30).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\cos^{10}\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} = \frac{1}{315e^{10}}[(128-616e^2+1179e^4-1126e^6+553e^8)E(e, \tfrac{1}{2}\pi)$$

$$-(128-680e^2+1479e^4-1691e^6+1069e^8-315e^{10})F(e, \tfrac{1}{2}\pi)] \ldots\ldots (31).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{[1-e^2\sin^2\theta]}\sin^8\theta d\theta = \frac{1}{315e^8}[(16+8e^2+12e^4+28e^6-64e^8)F(e, \tfrac{1}{2}\pi)$$

$$-(16+16e^2+21e^4+40e^6-128e^8)E(e, \tfrac{1}{2}\pi)] \ldots\ldots (32).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{[1-e^2\sin^2\theta]}\sin^6\theta\cos^2\theta d\theta = \frac{1}{315e^8}[2(8-4e^2-3e^4-4e^6+8e^8)E(e, \tfrac{1}{2}\pi)$$

$$-(16-16e^2-3e^4-5e^6+8e^8)F(e, \tfrac{1}{2}\pi)] \ldots\ldots (33).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{[1-e^2\sin^2\theta]}\sin^4\theta\cos^4\theta d\theta = \frac{1}{315e^8}[4(4-10e^2+6e^4+e^6-e^8)F(e, \tfrac{1}{2}\pi)$$

$$-(16-32e^2+9e^4+7e^6-8e^8)E(e, \tfrac{1}{2}\pi)] \ldots\ldots (34).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{[1-e^2\sin^2\theta]}\sin^2\theta\cos^6\theta d\theta = \frac{1}{315e^8}[2(8-28e^2+33e^4-10e^6+5e^8)E(e, \tfrac{1}{2}\pi)$$

$$-(16-64e^2+93e^4-50e^6+5e^8)F(e, \tfrac{1}{2}\pi)] \ldots\ldots (35).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{[1-e^2\sin^2\theta]}\cos^8\theta d\theta = \frac{1}{315e^8}[4(4-22e^2+51e^4-68e^6+35e^8)F(e, \tfrac{1}{2}\pi)$$

$$-(16-80e^2+165e^4-194e^6-35e^8)E(e, \tfrac{1}{2}\pi)] \ldots\ldots (36).$$

Thus we can proceed as far as we desire, finding the values of similar expressions almost without number.

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

120. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics in Boys' High School, Reading, Pa.

How many balls 1 inch in diameter can be put in a cubical box 1 foot in the *clear* each way, putting in the maximum number? [From Greenleaf's *Treatise on Algebra.*]

II. Solution by J. M. ARNOLD, Crompton, R. I.

In regard to the solution of Problem 120, given in the January number, it seems to me that the total number of balls, 2147, is not the maximum number that can be placed in the box.

Suppose we pack it this way: Place 144 balls on the bottom, then a layer of 121, and so on alternately until fifteen layers have been put in.

Now fill with sawdust so that the sawdust is just level with the tops of the balls. We have left a clear space of 12×12 and a little more than 1 inch in depth. Into this space can be placed 150 balls, by first placing a row of 12 balls against one side of the box, then a row of 11, and so on alternately until thirteen rows are put in; making 7 rows of 12 and 6 of 11 balls each, 150 in all.

The distance between the center lines of the rows will be $\frac{1}{2}\sqrt{3}$.

And $12\times\frac{1}{2}\sqrt{3}+1=11.3923$. $12-11.3923=.6077$ inch to spare.

For the whole box we have:

Eight layers of 144 each	=1152
Seven layers of 121 each	= 847
One layer of	150
Total	=2149

124. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.

At what time between 5 and 6 o'clock is the minute hand midway between 12 and the hour hand? When is the hour hand midway between 4 and the minute hand?

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Take a and $a+1$ as the hours between which the conditions are to happen.

Let x=the distance, after a o'clock, the minute hand must move.

Let $1\frac{1}{12}x$=the distance the hour hand moves.

(1). The first question of the problem involves two positions.

(a). In the first position, we readily find $5a+1\frac{1}{12}x=2x$.

$$\therefore\ x=\frac{60a}{23}.$$

(b). In the second position we find $5(12-a)-1\frac{1}{12}x=2(60-x)$.

$$\therefore\ x=\frac{60(12+a)}{23}.$$

Now put $a=5$. Then, in the first position, $x=13\frac{1}{23}$; and the time is $13\frac{1}{23}$ minutes past 5 o'clock.

In the second position, $x=44\frac{8}{23}$; and the time is $44\frac{8}{23}$ minutes past 5 o'clock.

(2). In the second question of the problem, take $4=b$.

Then, in general terms, we find

$$\frac{\pm(x-5b)}{2}=\pm[5(a-b)+\tfrac{1}{12}x].$$

$\therefore x=6(2a-b)$.

Plus of the double sign is to be used for $b<a$, and *minus* for $b>a$.

Put $a=5$, and $b=4$.

Then $x=36$, and the time is 36 minutes past 5 o'clock.

II. Solution by J. OWEN MAHONEY, B. E., M. Sc., Professor of Mathematics, Central High School, Dallas, Tex.; COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; and JENNIE B. BRIDENBAUGH, Student, Nebraska State Normal School, Peru, Neb.

Let x=the number of minute spaces over which the minute hand passes in each case. Then, in the first case, it is easily seen that

$$x+\tfrac{1}{12}x=25, \text{ or } x=13\tfrac{1}{23}.$$

Hence, at $13\frac{1}{23}$ minutes past 5 o'clock the first condition is satisfied.

In the second case,

$$x-2(\tfrac{1}{12}x+5)=20, \text{ or } x=36.$$

Hence, at 36 minutes past 5 o'clock the second condition is satisfied.

III. Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.; and J. T. FAIRCHILD, Principal, Crawfis College, Ohio.

A. 1. $\frac{2}{2}$=distance hour-hand moves past 5.

2. $\frac{24}{2}$=distance minute-hand moves in the same time.

3. $\frac{2}{2}+25$ minutes=distance from 12 to the hour-hand.

4. $\frac{1}{2}$ of ($\frac{2}{2}+25$ minutes)=$\frac{1}{2}+12\frac{1}{2}$ minutes=distance from 12 to the minute-hand, since it is to be half way between 12 and the hour-hand.

5. $\therefore \frac{24}{2}=\frac{1}{2}+12\frac{1}{2}$ minutes.

6. $\therefore \frac{23}{2}=12\frac{1}{2}$ minutes, and

7. $\frac{1}{2}=\frac{1}{23}$ of $12\frac{1}{2}$ minutes=$\frac{23}{46}$ minutes, and

8. $\frac{24}{2}=24$ times $\frac{23}{46}$ minutes=$13\frac{1}{23}$ minutes, the time past 5, when the minute-hand will be midway between 12 and the hour-hand.

B. 1. $\frac{2}{2}$=distance hour-hand moves past 5.

2. $\frac{2}{2}+5$ minutes=distance from 4 to hour-hand.

3. $\frac{4}{2}$+10 minutes=distance from 4 to minute hand, since hour-hand is half way between 4 and the minute-hand.

4. $\frac{4}{2}$+30 minutes=distance from 12 to minute hand.

5. $\therefore \frac{24}{2}=\frac{4}{2}+30$ minutes.

6. $\therefore \frac{20}{2}=30$ minutes.

7. $\frac{1}{2}=\frac{1}{20}$ of 30 minutes=$1\frac{1}{2}$ minutes.

8. $\frac{24}{2}$=24 times $1\frac{1}{2}$ minutes=36 minutes=time past 5.

IV. Solution by Hon. JOSIAH H. DRUMMOND, Portland, Me.; ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Ia.; W. MANZILLA, Instructor in Mathematics, Langston University, Langston, Okla., and JOHN M. COLAW, A. M., Monterey, Va.

As the minute-hand travels over twelve minute-spaces while the hour-hand is going over one, the distance from XII to the hour-hand is $25+\frac{1}{12}$ of the distance the minute-hand travels; and one-half of this is $\frac{25}{2}+\frac{1}{24}$ of that distance.

Hence, $\frac{25}{2}$ is $\frac{23}{24}$ of that distance which is $13\frac{1}{23}$, and the time is $13\frac{1}{23}$ minutes past five.

Second. Of course at two o'clock the hour-hand is precisely half-way between IV and the minute-hand.

Again, while the minute-hand travels from XII to IV, the hour-hand goes over $\frac{20}{12}$ minute-spaces. Hence $\frac{20}{12}+\frac{1}{12}$ of the distance the minute-hand travels after it reaches IV=one-half of that distance.

Hence $\frac{20}{12}$ is $\frac{5}{12}$ of that distance which is four. And the time required is twenty-four minutes past four.

ALGEBRA.

99. Proposed by CHARLES HALLETTE JUDSON, Professor of Mathematics, Furman University, Greenville, S. C.

Seven persons met at a summer resort, and agreed to remain as many days as there are ways of sitting at a round table, so that no one shall sit twice between the same two companions. They remained fifteen days. It is required to show in what way they may have been seated.

I. Solution by P. H. PHILBRICK, C. E., Chief Engineer for Kansas City, Watkins & Gulf Railway Co., Lake Charles, La.

Each person has six companions; and the the number of ways of sitting are, therefore, equal to the number of ways of selecting two out of six or 6.5/1.2 =15 ways.

II. Solution by the PROPOSER.

I give below the seating of 6 persons 10 ways, and of 8 persons 21 ways, but I have failed to get a correct seating of 7.

n persons can be seated in $\frac{(n-1)(n-2)}{2}$ ways.

1	A^1	B^2	C^3	D^4	E^5	F^6
2	A^1	B^2	D^4	C^3	F^6	E^5
3	A^1	B^2	E^5	F^6	D^4	C^3
4	A^1	B^2	F^6	E^5	C^3	D^4
5	A	C	F	D	B	E
6	A	C	E	B	F	D
7	A	C	B	E	D	F
8	A	D	E	C	B	F
9	A	D	B	F	C	E
10	A	E	D	B	C	F

1	A	B	C	D	E	F	G	H
2	A	B	D	F	G	C	H	E
3	A	B	E	G	F	H	C	D
4	A	B	F	C	H	D	E	G
5	A	B	G	H	C	E	D	F
6	A	B	H	E	D	G	F	C
7	A	C	G	E	H	F	B	D
8	A	C	D	H	E	B	F	G
9	A	C	E	F	B	G	D	H
10	A	C	H	B	F	D	G	E
11	A	C	B	D	G	H	E	F
12	A	D	H	G	E	C	B	F
13	A	D	F	E	G	B	C	H
14	A	D	G	C	B	H	F	E
15	A	D	E	B	C	F	H	G
16	A	E	C	F	D	H	B	G
17	A	E	D	B	H	G	C	F
18	A	E	B	G	C	D	F	H
19	A	F	H	D	B	E	C	G
20	A	F	G	B	D	C	E	H
21	A	G	D	C	F	E	B	H

In seating the 8 persons, the first six arrangements are quite simple, as indicated. Now take 6 reversed, and permute them just as the first 6. This gives the next 5. Take any other, as 3, reversed, and permute in same way. Take 4 in same way rejecting first 1, then 2, then 3, then 4, which have been previously arranged, and we finally get 21.

[Neither of the above solutions show *how* to seat 7 persons in the 15 ways. For the first correct arrangement sent us, we are authorized to offer a year's subscription to the MONTHLY. EDITOR.]

100. Proposed by W. H. CARTER, Vice President, and Professor of Mathemetics, Centenary College, Jackson, La.

Solve, $x^{x-y}=y^{4a}$, $y^{x+y}=x^a$.

I. Solution by CHARLES C. CROSS, Meredithville, Va.; P. S. BERG, A. M., Larimore, N. D.; J. D. CRAIG, Frankfort, N. J.; JOSIAH H. DRUMMOND, LL. D., Portland, Me.; ALOIS F. KOVARIK, Decorah, Ia.; P. H. PHILBRICK, C. E., Lake Charles, La.; and COOPER D. SCHMITT, A. M., Knoxville, Tenn.

Taking logarithms, we have, $(x+y)\log x=4a\log y$.....(1);

and $(x+y)\log y=a\log x$ .....(2).

(1) divided by (2) gives $\log^2 x=4\log^2 y$,

or $\log x=2\log y$.....(3);

Therefore $x=y^2$ .....(4).

Substituting (3) in (2), and dividing by $\log y$,

we get $x+y=2a$ .....(5).

Substituting (4) in (5), $y^2+y=2a$.

Therefore, $y=\frac{1}{2}[-1\pm\sqrt{}(8a+1)]$.

$\therefore x=\frac{1}{2}[4a+1\mp\sqrt{}(8a+1)]$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; J. SCHEFFER, A. M., Hagerstown, Md.; J. W. YOUNG, Fellow and Assistant in Mathematics, Ohio State University, Columbus, O.; ELMER SCHUYLER, B. Sc., Professor of German and Mathematics in Boys' High School, Reading, Pa.

$(x+y)\log x=4a\log y$ ..... (1);

$(x+y)\log y=a\log x$ .....(2).

(1)÷(2) gives $(\log x)^2=4(\log y)^2$.

$\therefore \log x=\pm 2\log y$. $\therefore x=y^2$ or $1/y^2$.

$\therefore 2(y^2+y)=4a$. $\therefore y^2+y=2a$.

$\therefore y=-\frac{1}{2}[1\pm\sqrt{}(1+8a)]$; $x=\frac{1}{2}[1+4a\mp\sqrt{}(1+8a)]$.

When $x=1/y^2$, we get $y^3+2ay^2=-1$.

The roots of this equation can be found by Cardan's Rule.

[ZERR, SCHEFFER]

Rational values are obtained by substituting for a the triangular numbers 1, 3, 6, 10, 15, 21, 28, etc. The equation $y^{-2}+y+2a=0$ leads to the cubic $y^3+2ay^2+1=0$. In case of $a=3$, we have the simultaneous sets

$y=2$; $y=-3$; $y=-6.0275$;

$x=4$; $x=9$; $x=-.0275$;

and two imaginary roots. [SCHEFFER.]

III. Solution by WALTER HUGH DRANE, Graduate Student, Harvard University, Cambridge, Mass.; and CHAS. E. MYERS, Canton, Ohio.

$x^{x+y}=y^{4a}$ ..... (1). $y^{x+y}=x^a$ .....(2).

From (1) $y=x^{(x+y)/4a}$, and from (2) $y=x^{a/(x+y)}$.

$\therefore x^{(x+y)/4a}=x^{a/(x+y)}$ .....(3). $\therefore (x+y)/4a=a/(x+y)$ ..... (4).

$\therefore x+y=\pm 2a$ .....(5).

This in (1) gives, taking + value of a, $x^{2a}=y^{4a}$.

$\therefore x=y^2$. $\therefore$ By (2) $y^{2+y}=y^{2a}$. $\therefore y^2+y=2a$.

$\therefore y=\dfrac{-1\pm\sqrt{}(1+8a)}{2}$; $\therefore x=\dfrac{1+4a\mp\sqrt{}(1+8a)}{2}$.

[DRANE, MYERS.]

Taking the negative value of $2a$ leads us to the cubic $y^3+2ay^2+1=0$, and $x=1/y^2$. Substitute $z=y+2a/3$, and this reduces to $z^3-4a^2z+1=0$. Solving this by Cardan's method gives

$$u=\tfrac{1}{2}[-1\pm\sqrt{}(1-\tfrac{256}{27}a^6)], \text{ and } v=\tfrac{1}{2}[-1\mp\sqrt{}(1-\tfrac{256}{27}a^6)],$$

where $u^{\frac{1}{3}}+v^{\frac{1}{3}}=z$, and $u^{\frac{1}{3}}v^{\frac{1}{3}}=4a^2/3$. $\therefore$ Three values of z are,

$$z_1=\left(\frac{-1\pm\sqrt{(1-\frac{2^5}{2^7}a^6)}}{2}\right)^{\frac{1}{3}}+\frac{4a^2\sqrt[3]{2}}{[-1\pm\sqrt{(1-\frac{2^5}{2^7}a^6)}]^{\frac{1}{3}}};$$

$$z_2=w\left(\frac{-1\pm\sqrt{(1-\frac{2^5}{2^7}a^6)}}{2}\right)^{\frac{1}{3}}+\frac{4a^2\sqrt[3]{2}}{w[-1\pm\sqrt{(1-\frac{2^5}{2^7}a^6)}]^{\frac{1}{3}}};$$

$$z_3=w^2\left(\frac{-1\pm\sqrt{(1-\frac{2^5}{2^7}a^6)}}{2}\right)^{\frac{1}{3}}+\frac{4a^2\sqrt[3]{2}}{w^2[-1\pm\sqrt{(1-\frac{2^5}{2^7}a^6)}]^{\frac{1}{3}}}.$$

$\therefore$ $y_1=z_1-2a/3$, $y_2=z_2-2a/3$, and $y_3=z_3-2a/3$, are the three required values of y; and $x_1=1/y_1^2$, $x_2=1/y_2^2$, $x_3=1/y_3^2$, are the three values of x. It is easily seen that the values of y, and hence of x, are all real, one real and two imaginary, or real and multiple, according as $a^6>$, $<$, or $=\frac{2^7}{2^5 6}$. [DRANE.]

GEOMETRY.

122. Proposed by G. I. HOPKINS, A. M., Professor of Mathematics and Physics, Manchester High School, Manchester, N. H.

If perpendiculars are dropped from the vertices of a regular polygon upon any diameter of the circumscribed circle, the sum of the perpendiculars which fall on one side of this diameter is equal to the sum of those which fall on the opposite side. [From Chauvenet's *Treatise on Elementary Geometry*.]

III. Solution by WILLIAM H. ECHOLS, B. S., C. E., Professor of Mathematics in University of Virginia, Charlottesville, Va.

A very simple proof of this is:

1°. Let there be a regular polygon of $2n$ sides, p_1, $\ldots\ldots p_{2n}$, the perpendiculars from vertices on any straight line.

Then if p_c is perpendicular from the center, clearly

$$\sum_1^{2n} p_r=2np_c.$$

2°. Form a polygon of n sides by joining the mid-points of alternate sides of 1°. Then at once

$$2\sum_1^{n} p'_r=\sum_1^{2n} p_r=2np_c,$$

where p'_r are perpendiculars of n-side.

This is true whatever integer be n.

3°. If the line of projection passes through the center, the theorem is demonstrated generally.

125. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

To find the locus of a point on the surface of an ellipsoid which has the property that the tangent plane at that point is at the given distance, f, from the center of the ellipsoid.

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The equation to the tangent plane to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \ldots\ldots (1)$$

at the point (x', y', z') is

$$\frac{x'x}{a^2}+\frac{y'y}{b^2}+\frac{z'z}{c^2}=1 \ \ldots\ldots (2).$$

Then

$$\frac{1}{f^2 g}=\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4} \ldots\ldots (3),$$

is the required locus, an ellipsoid concentric with (1).

Also solved by *G. B. M. ZERR*, and *ELMER SCHUYLER*.

126. Proposed by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

Through any fixed point O draw two straight lines at right angles. Let one line cut a given circle at Q, the other at R. Find, by Euclidean methods, the locus of the foot of the perpendicular from O upon the chord QR. Give complete analysis and discussion. Solve also by coördinate geometry.

I. Solution (Analytical) by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Let r=the radius of the given circle, a=the distance of its center from O, and take the line through O and the center for the axis of x. Then the coördinate axes being rectangular, the equation to the fixed circle is

$$(x-a)^2+y^2=r^2 \text{ or } x^2+y^2-2ax+a^2-r^2=0 \ldots\ldots (1).$$

If $lx+my=1$ .....(2) be the equation to QR, the equation to the lines OQ and OR is

$$x^2+y^2-2ax(lx+my)+(a^2-r^2)(lx+my)^2=0 \ldots\ldots (3), \text{ or}$$

$$[1-2al+(a^2-r^2)l^2]x^2+2[lm(a^2-r^2)-am]xy+[1+(a^2-r^2)m^2]y^2=0 \ \ldots\ldots (4).$$

The condition that these lines are perpendicular to each other is

$$2-2al+(a^2-r^2)(l^2+m^2)=0 \ldots\ldots (5).$$

The line through O and perpendicular to (2) is $x/y=l/m$ .....(6).

Making (5) homogeneous by aid of (2), we have,

$$[2x^2-2ax+(a^2-r^2)]l^2+(4xy-2ay)lm+[2y^2+(a^2-r^2)]m^2=0 \ldots\ldots (7).$$

l and m by (6) being proportional to x and y, (7) easily becomes

$$2x^4-2ax^3+(a^2-r^2)x^2+4x^2y^2-2axy^2+2y^4+(a^2-r^2)y^2=0, \text{ or}$$

$$2(x^2+y^2)^2-[2ax-(a^2-r^2)](x^2+y^2)=0 \ldots\ldots (8),$$

giving the two circles

$$x^2+y^2=0 \ldots\ldots (9), \quad x^2+y^2-ax+\tfrac{1}{2}(a^2-r^2)=0 \ldots\ldots (10)$$

for the required loci.

II. Solution by J. W. YOUNG, Fellow and Assistant in Mathematics, Ohio State University, Columbus, O.; and J. SCHEFFER, A. M., Hagerstown, Md.

To solve the problem by coördinate geometry, we may proceed as follows: Let the fixed point be the origin, and let the axis of x pass through the center of the circle. Its equation is

$$x^2+y^2+2gx+c=0 \ldots\ldots (1).$$

Let the chord, QR, be $y=mx+a \ldots\ldots (2)$.

Make (1) homogeneous by means of the relation $(y-mx)/a=1$. The resulting equation, viz.,

$$x^2+y^2+2gx\frac{(y-mx)}{a}+\frac{c(y-mx)^2}{a^2}=0 \ldots\ldots (3),$$

represents the lines through O, and Q and R.

(3) may be written $(a^2-2agm+cm^2)x^2+(a^2+c)y^2+2(ag-cm)xy=0$.

But the lines OQ and OR are at right angles. The condition for this is

$$(a^2-2agm+cm^2)+(a^2+c)=0, \text{ or } 2a^2+cm^2-2agm+c=0 \ldots\ldots (4).$$

The perpendicular from O and QR is $y=-(1/m)x \ldots\ldots (5)$.

We must find the locus of the intersection of (2) and (5), under the condition (4). (5) gives $m=-x/y$. (2) gives $a=y-mx=(y^2+x^2)/y$.

Substitute for m and a in (4), and we have

$$2\frac{(y^2+x^2)^2}{y^2}+c\left(\frac{x^2}{y^2}+1\right)+2g\frac{(x^2+y^2)x}{y^2}=0, \text{ or } 2(x^2+y^2)+2gx+c=0,$$

the required locus. This is evidently a circle, with center half way between O and the center of the given circle.

NOTE ON PROBLEM 118.

BY GEORGE R. DEAN, ROLLA, MO.

The analysis of a geometrical problem has always appeared to me more interesting and useful than the mere construction and proof.

Any problem in maxima and minima may be solved by applying the general principle that a maximum or a minimum value of a variable quantity lies between two equal values. Let P be any point in the line FH. At some other point Q the angle DQC is equal to the angle DPC. The point Q is obviously found by drawing a circle through D, C, and P. The point at which the angle is a maximum is situated between P and Q. It is obvious that if the point P had been so selected that Q would coincide with it, the required point would be determined. A circle drawn through D and C tangent to the given line will cut the line in two coincident points between which the maximum point lies, and is therefore coincident with them.

As an example of this kind of analysis I propose the following problem:

Find a point in a given line such that the sum of its distances from two fixed points is a minimum. Give the analysis.

CALCULUS.

94. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

Find the minimum isosceles triangle that can be described about a given ellipse, having its base parallel to the major axis.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; M. C. STEVENS, A. M., Professor of Higher Mathematics, Purdue University, Lafayette, Ind.; and H. C. WHITAKER, M. E., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

Let x, y be the coördinates of P. Then $x^2/a^2+y^2/b^2=1$ is the equation to the ellipse.

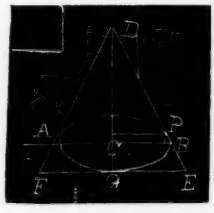
$CD=b^2/y$. $\therefore GD=b^2/y+b=b(b+y)/y$.

$AC=a^2/x$. $\therefore FG=a^2(b+y)/bx$.

$\therefore a^2(b+y)^2/xy=$area$=$minimum.

$\therefore dy/dx=y(y+b)/x(y-b)$.

Also $dy/dx=-b^2x/a^2y$, from the equation to the ellipse.



$$\therefore \frac{b^2x}{a^2y}=\frac{y(b+y)}{x(b-y)}, \text{ or } x^2=\frac{a^2y^2(b+y)}{b^2(b-y)}.$$

This value of x^2 in the equation to the ellipses gives, after reduction, $2y^2+by=b^2$.

$\therefore y=\frac{1}{2}b$ for a minimum.

$\therefore$ Altitude$=b(b+y)/y=3b$. Base$=2a^2(y+b)/bx=3a^2/x=2a\sqrt{3}$.

Side$=\sqrt{(9b^2+3a^2)}$. Area$=3ab\sqrt{3}$.

III. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Let the equation to tangent DE be $(x/a)\cos\phi+(y/b)\sin\phi=1$.
Then $CB=a\sec\phi$; $CD=b\operatorname{cosec}\phi$. $DG=b+b\operatorname{cosec}\phi$.
Area$=DG.GE$. $GE : CB :: DG : DC$. Whence

$$GE=\frac{a\sec\phi(1+\operatorname{cosec}\phi)}{\operatorname{cosec}\phi}=\frac{a\sec\phi(1+\operatorname{cosec}\phi)}{\operatorname{cosec}\phi}b(1+\operatorname{cosec}\phi)=ab\tan\phi(1+\operatorname{cosec}\phi)^2$$

Hence $\tan\phi(1+\operatorname{cosec}\phi)^2$ is to be examined for a minimum. Differentiating and equating to zero, we have

$$\sec^2\phi(1+\operatorname{cosec}\phi)^2-2\tan\phi(1+\operatorname{cosec}\phi)\operatorname{cosec}\phi\cot\phi=0,$$

whence, $\sec^2\phi(1+\operatorname{cosec}\phi)=2\operatorname{cosec}\phi$, or

$$\frac{1+\sin\phi}{\sec\phi\cos^2\phi}=\frac{2}{\sin\phi}.$$

Solving, $\sin\phi=\frac{1}{2}$; whence $\operatorname{cosec}\phi=2$.
Hence $DG=b+b\operatorname{cosec}\phi=b+2b=3b$.

Also solved by *P. S. BERG*, *W. H. DRANE*, *J. SCHEFFER*, and *J. W. YOUNG*.

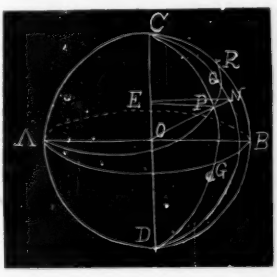
95. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

A ship starts at the equator and sails northeast at all times. How far has the ship sailed (in miles) when her latitude is 30°, 45°, 60°, 90°? How far when her longitude is 90°, 180°, 270°, 360°? Regarding the earth as a sphere, radius 3956 miles.

Solution by the PROPOSER.

Let A be the point of the ship's departure, $APQR$ the ship's course, P, Q two consecutive points on the course, $AG=\theta=$longitude of P, $PG=\phi=$latitude of P, $OP=r=$radius of the earth, $\angle PQN=\beta=\frac{1}{4}\pi$.

Then $PQ=ds$, $PN=EP\times\angle PEN=r\cos\phi d\theta$, $QN=rd\phi$.
$\therefore\ ds^2=r^2(\cos^2\phi d\theta^2+d\phi^2)$.
$\therefore\ ds=r(\cos^2\phi d\theta^2+d\phi^2)^{\frac{1}{2}}$ .....(1).
$PN/QN=\tan\beta=\cos\phi d\theta/d\phi$.
$\therefore\ d\theta=\tan\beta d\phi/\cos\phi$.....(2).
(2) in (1) gives $ds=\sqrt{(1+\tan^2\beta)}d\phi=rd\phi/\cos\beta$.
$\therefore\ s=(r/\cos\beta)\int_0^{\Phi}d\phi=r\phi/\cos\beta=r\phi\sqrt{2}$.



When $\phi=\frac{1}{6}\pi$, $s=\frac{1}{6}\pi r\sqrt{2}=2929.3817$ miles.
When $\phi=\frac{1}{4}\pi$, $s=\frac{1}{4}\pi r\sqrt{2}=4394.07257$ miles.
When $\phi=\frac{1}{3}\pi$, $s=\frac{1}{3}\pi r\sqrt{2}=5858.7634$ miles.
When $\phi=\frac{1}{2}\pi$, $s=\frac{1}{2}\pi r\sqrt{2}=8788.14514$ miles.

From (2) $d\phi = \cos\phi d\theta / \tan\beta$ ..... (3).
(3) in (1) gives

$$ds = r\cos\phi \,/\, (1+\tan^2\beta)d\theta/\tan\beta = r\cos\phi d\theta/\sin\beta.$$

From (2), $\theta = \tan\beta \int \frac{d\phi}{\cos\phi} = \tan\beta \log[\tan(\frac{1}{4}\pi + \frac{1}{2}\phi)]$.

$$\therefore e^{\theta\cot\beta} = \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi) = \frac{1+\sin\phi}{\cos\phi}. \quad \therefore \cos\phi = \frac{2}{e^{\theta\cot\beta} + e^{-\theta\cot\beta}} = \frac{2}{e^{\theta} + e^{-\theta}},$$

since $\cot\beta = 1$.

$$\therefore s = 2r \,/\, 2\int_0^{\theta} \frac{d\theta}{e^{\theta} + e^{-\theta}} = 2r \,/\, 2(\tan^{-1}e^{\theta} - \tfrac{1}{4}\pi).$$

When $\theta = \frac{1}{2}\pi$, $s = 2r \,/\, 2(\tan^{-1}e^{\frac{1}{2}\pi} - \frac{1}{4}\pi) = r \,/\, 2(.3695185\pi) = 6494.764423$ miles.
When $\theta = \pi$, $s = 2r \,/\, 2(\tan^{-1}e^{\pi} - \frac{1}{4}\pi) = r \,/\, 2(.472506\pi) = 8304.902620$ miles.
When $\theta = \frac{3}{2}\pi$, $s = 2r \,/\, 2(\tan^{-1}e^{\frac{3}{2}\pi} - \frac{1}{4}\pi) = r \,/\, 2(.494281\pi) = 8687.626341$ miles.
When $\theta = 2\pi$, $s = 2r \,/\, 2(\tan^{-1}e^{2\pi} - \frac{1}{4}\pi) = r \,/\, 2(.498804\pi) = 8767.1239018$ miles.

Also solved by *J. SCHEFFER*. A somewhat different solution of this problem is given in Finkel's *Mathematical Solution Book*, page 344.

96. Proposed by **W. H. CARTER**, Vice President, and Professor of Mathemetics, Centenary College, Jackson, La.

If $f(x) = \int f(x)dx$, find $f(x)$, the constant being zero.

Solution by **W. F. SHAW**, 1600 Sabine Street, Austin, Tex.

$$f(x) = \int f(x)dx. \quad df(x) = f(x)dx.$$

$$\frac{df(x)}{f(x)} = dx. \quad \log f(x) = x. \quad f(x) = e^{x}.$$

Also solved by *W. H. DRANE*, *J. SCHEFFER*, and *G. B. M. ZERR*.

MECHANICS.

87. Proposed by **H. C. WHITAKER**, M. E., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

> "He on his impious foes onward drove,
> Drove them before him to the bounds
> And crystal walls of Heaven; which opening wide
> Rolled inward and a spacious gap disclosed
> Into the wasteful deep; headlong themselves they threw
> Down from the verge of Heaven.
> Nine days they fell; Hell at last
> Yawning received them whole and on them closed!"
>
> *Paradise Lost, Book VI.*

Assuming Hell to be the center of the earth, and the only force acting on the lost spirits to be that of gravity due to the earth's attraction,—How far is Heaven ?

Solution by P. H. PHILBRICK, C. E., Chief Engineer for Kansas City, Watkins & Gulf Railway Co., Lake Charles, La.

Let a be the distance from the center of the earth, or origin. g the force of gravity at the surface. Then the equation outside the earth is, $dx^2/dt^2 = -gr^2/x^2$. Multiply by $2dx$, integrate, and we find,

$$\frac{dx^2}{dt^2} = v^2 = 2gr^2\left(\frac{1}{x} - \frac{1}{a}\right) \ldots\ldots (1).$$

At the surface of the earth $x=r$, and we have from (1),

$$v = 2gr\left(\frac{a-r}{a}\right)^{\frac{1}{2}} = \sqrt{2gr}, \text{ very nearly,} \quad \ldots\ldots (2),$$

since a is very great compared with r.

The force of gravity inside the earth and distant x from the center is, $g(x/r) = (g/r)x$.

Hence the equation of motion is, $d^2x/dt^2 = -(g/r)x$.

Integrating we have, $dx^2/dt^2 = (g/r)(r^2 - x^2)$, since the "lost spirits," so far as this force is concerned, are supposed at rest when $x=r$ and $t=0$.

From above,

$$\sqrt{\frac{g}{r}}\,dt = \frac{-dx}{\sqrt{(r^2-x^2)}}. \quad \therefore\ t = \sqrt{\frac{r}{g}}\cos^{-1}\frac{x}{r} \ldots\ldots (3).$$

From (2) we have

$$\frac{r}{v} = \sqrt{\frac{r}{2g}}\left(\frac{a}{a-r}\right)^{\frac{1}{2}} = \sqrt{\frac{r}{2g}}, \text{ very nearly} = t' \ldots\ldots (4).$$

This gives the time of passing from the surface to the center of the earth by virtue of the velocity at the surface.

Let $r = 20880000$ and $g = 32$, and (3) and (4) become,

$$t = 808\cos^{-1}\frac{x}{r} \quad \ldots\ldots (3'), \text{ and } t' = \sqrt{\frac{r}{2g}} = 571.2 \quad \ldots\ldots (4').$$

We may find from these equations by trial, the time required for the spirits to pass from the surface to the center of the earth.

Thus in (3') let $x = .83195a$.

Then $t = 808\cos^{-1}.83195 = 808 \times .588176 = 475.246$, and $t' = .83195 \times 571.2 = 475.21$.

About $\frac{5}{6}$ of the radius of the earth is traversed by virtue of the velocity at the surface, and the remaining $\frac{1}{6}$ by virtue of the interior attractive force.

Nine days $=9\times24\times60\times60=777600$ seconds
Deduct 475 seconds

$\therefore$ Time of falling to surface of the earth $=777125$ seconds.

The corresponding distance is, $\frac{1}{2}gt^2=16\frac{1}{12}(777125)^2=9713099188802$ feet $=1839602119$ miles.

Also solved by *H. C. WHITAKER, G. B. M. ZERR, B. F. SINE, J. SCHEFFER, W. H. DRANE,* and *J. B. GREGG.* Mr. Gregg furnished a very elaborate solution, indicating the various steps of the computation, his result being 360733 miles. Dr. Whitaker gets the same result. The results of the various contributors differ widely, due to variously assumed values of the constants, and, in some cases, considering the earth a mere point.

95. Proposed by **FLORIAN CAJORI**, Ph. D., Author of History of Mathematics, History of Physics, etc., and Professor of Mathematics, Colorado College, Colorado Springs, Col.

Assuming that the velocity is proportionate to the distance described from the state of rest, (1) can the body start in motion? (2) If it can, what is its initial acceleration? If we make the additional assumption that the time of fall, from rest, through a finite distance is finite, does it follow that the distance is infinite?

Solution by **WALTER H. DRANE**, Graduate Student, Harvard University, Cambridge, Mass.

1. The statement here is somewhat confusing. No body can start in motion unless acted upon by an external force; if it be meant here then to ask, can the body be started, the answer is self-evident. Theoretically, an infinitesimal force would be sufficient to put the body in motion though the time might become infinite before the velocity became finite.

2. We have $ds/dt=ks$ where k is a constant depending upon initial conditions. Differentiating we get $d^2s/dt^2=k(ds/dt)=k^2s$; that is, the initial acceleration is k time the initial velocity and is itself proportional to the distance. What the value of this initial acceleration is, depends upon the external force acting and the mass of the body.

3. Nothing is said here about the initial force acting. If we assume it infinitesimal, it follows from 1, that if the time of fall through a finite distance is finite, then the velocity must be infinite.

DIOPHANTINE ANALYSIS.

78. Proposed by **COOPER D. SCHMITT**, A. M., Professor of Mathematics in University of Tennessee, Knoxville, Tenn.

Find three square numbers in harmonical progression.

I. Solution by **M. A. GRUBER**, A. M., War Department, Washington, D. C.

The terms of an harmonical progression are the reciprocals of such numbers as form an arithmetical progression.

Let a^2, b^2, and c^2 be three square numbers in arithmetical progression, $a<b<c$. Then $b^2-a^2=c^2-b^2$, or $a^2+c^2=2b^2$. a, b, and c are rendered rational and integral by putting $a=p^2-q^2$ the difference between $2pq$, $b=p^2+q^2$, and $c=p^2-q^2+2pq$.

$\therefore$ The required harmonical progression is

$$1/(p^2-q^2-2pq)^2,\ 1/(p^2+q^2)^2,\ 1/(p^2-q^2+2pq)^2,$$

in which p and q are any integers, $p>q$.

Put $p=2$ and $q=1$. We then have $1/1^2$, $1/5^2$, $1/7^2$.

Put $p=3$ and $q=2$. We then have $1/7^2$, $1/13^2$, $1/17^2$.

II. Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

If a^2, b^2 are two numbers, then will a^2, $2a^2b^2/(a^2+b^2)$, b^2 be in harmonical progression. Let $a=m-n$, $b=m+n$; therefore, $(m-n)^2$, $(m^2-n^2)^2/(m^2+n^2)$, $(m+n)^2$ will be the required numbers, if $m^2+n^2=\square$.

Let $m=p^2-q^2$, $n=2pq$.

$\therefore (q^2+2pq-p^2)^2$, $(p^4-6p^2q^2+q^4)^2/(p^2+q^2)^2$, $(p^2+2pq-q^2)^2$ are the numbers required.

Let $p=2$, $q=1$. $\therefore$ 1, $\frac{49}{25}$, 49 are the numbers.

Let $a=m$, $b=7m$. Then m^2, $49m^2/25$, $49m^2$ are the numbers required.

Let $m=5n$. $\therefore$ $25n^2$, $49n^2$, $1225n^2$ are the required numbers. This gives a series of whole numbers by giving n integral values.

III. Solution by JOSIAH H. DRUMMOND, LL. D., Portland, Me.; A. H. BELL, Hillsboro, Ill.; and CHAS. C. CROSS, Meredithville, Va.

As the reciprocal of numbers in arithmetical progression are in harmonical progression, we may solve the question by obtaining three squares in arithmetical progression.

Let $x^2-2xy+y^2$, x^2+y^2, and $x^2+2xy+y^2$, be the three squares, and it only remains to make $x^2+y^2=\square$. This is so when $x=p^2-q^2$ and $y=2pq$; then $x^2+y^2=(p^2+q^2)^2$ in which p and q may be any unequal numbers.

Take $p=2$, and $q=1$, and the squares are 1, 25, and 49, and the reciprocals are 1, $\frac{1}{25}$, and $\frac{1}{49}$, or if integrals are desired, 1225, 49, and 25.

Also solved by *O. S. WESTCOTT, SYLVESTER ROBINS, ALOIS F. KOVARIK, J. W. YOUNG*, and the *PROPOSER*.

79. Proposed by EDMUND FISH, Hillsboro, Ill.

Find an integral right triangle in which the bisector of one of the acute angles is also integral.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

In any right triangle, let a=altitude, b=base, c=hypotenuse, c_1=bisector of angle opposite base, c_2=bisector of angle opposite altitude.

Put $a=tpq$, $b=t(p^2-q^2)/2$, and $c=t(p^2+q^2)/2$; t, p, and q being any integers, $p>q$.

Then from solution I, problem 43, page 95, Vol. IV, No. 3, THE AMERICAN MATHEMATICAL MONTHLY, we find

$$c_1=\frac{tpq\sqrt{[2(p^2+q^2)]}}{p+q}, \text{ and } c_2=\frac{t(p^2-q^2)\sqrt{(p^2+q^2)}}{2p}.$$

c_2 will be rational and integral when $p^2+q^2=\square$, and $t=2p$. The general values for p and q, in this case, are $2mn$ and m^2-n^2, $m>n$, $p>q$.

Put $m=2$ and $n=1$. Then $p=4$, $q=3$, and $t=8$.

Whence, $a=96$, $b=28$, $c=100$, and $c_2=35$.

Put $m=3$, and $n=2$. Then $p=12$, $q=5$, and $t=24$.

Whence, $a=1440$, $b=1428$, $c=2028$, and $c_2=1547$.

c_1 will be rational and integral when $p^2+q^2=2\times\square$, and $t=\frac{1}{2}(p+q)$.

The general values for p and q, in this case, are m^2-n^2+2mn and m^2-n^2 the difference between $2mn$, $m>n$ and $p>q$. The numerical results are the same as those obtained in case of c_2, except that a and b have interchanged values.

Of course, only one of the bisectors can be rational in any one triangle.

II. Solution by A. H. BELL, Hillsboro, Ill.

Let the hypotenuse$=h$, and the two legs$=a$ and $2c$, bisector$=b$, and segments $c+s$ and $c-s$.

Then $ah=(c+s)(c-s)+b^2$. $\therefore b^2=ah-(c^2-s^2)$ .... (1).

$a(c+s)=h(c-s)$. $\therefore s=c(h-a)/(h+a)$ .....(2).

Substituting the general value of the sides of a right triangle in (1) and (2), m^2+n^2, m^2-n^2, $2mn$; $s=n^3/m$, $s^2=n^6/m^2$.

$b^2=(m^2+n^2)(m^2-n^2)^2/m^2=\square$, if m and n are taken for the legs of any integral right triangle. When all the parts are multiplied by m, we obtain integral values, and they are $m(m^2+n^2)$, $m(m^2-n^2)$, and $2m^2n$ for the sides; $\sqrt{[(m^2+n^2)](m^2-n^2)}$=bisector; $n(m^2\pm n^2)$=segments.

In the right triangle 5, 4, 3, take m and $n=4$ and 3; then the required right triangle is 100, 28, 96; bisector=35, segments=75 and 21. In the right triangle 13, 12, 5, m and $n=12$ and 5; then the required sides=2028, 1428 and 1440, bisector=1547, segments=845 and 595.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let ABC be a right triangle, right angled at C, AD the bisector of $\angle A$.

Let $BC=a$, $AC=b$, $AB=c$, $AD=d$.

Then $\frac{1}{2}cd\sin\frac{1}{2}A+\frac{1}{2}bd\sin\frac{1}{2}A=\frac{1}{2}bc\sin A$. $cd+bd=2bc\cos\frac{1}{2}A$.

$\therefore d=2bc\cos\frac{1}{2}A/(b+c)$. Also $d=b\sec\frac{1}{2}A$.

$\therefore \sec\frac{1}{2}A=\sqrt{[2c/(b+c)]}$. $\therefore d=b\sqrt{[2c/(b+c)]}$.

Let $a=2pq$, $b=p^2-q^2$, $c=p^2+q^2$. $\therefore \sqrt{[2c/(b+c)]}=\sqrt{(p^2+q^2)/p}$.

Let $p=m^2-n^2$, $q=2mn$. $\therefore \sqrt{[2c/(b+c)]}=\sqrt{(p^2+q^2)/p}=(m^2+n^2)/(m^2-n^2)$.

Let $m=3r$, $n=r$. Then $p=8r^2$, $q=6r^2$. $\therefore a=96r^4$, $b=28r^4$, $c=100r^4$, $d=35r^4$, where r can have any integral value.

IV. Solution by O. S. WESTCOTT, A. M., Sc. D., Principal North Division High School, Chicago, Ill.

$BD : DC :: x^2+y^2 : x^2-y^2$, and $BD+DC$ or $BC : DC :: 2x^2 : x^2-y^2$.

$\therefore$ $DC=2xy(x^2-y^2)/2x^2=[(x^2-y^2)/x]y$.

And $AD^2=(x^2-y^2)^2+y^2[(x^2-y^2)^2/x]y^2$.

$AD^2=(x^2+y^2)(x^2-y^2)^2/x^2$. $AD=[(x^2-y^2)/x]\sqrt{(x^2+y^2)}$.

Put $x=4$, $y=3$, and $AD=8\frac{4}{5}$. Increasing each dimension of triangle in quadruple ratio $AD=35$, $AC=28$, $BC=96$, $AB=100$, $DC=21$.

Also solved by *CHAS. C. CROSS, JOSIAH H. DRUMMOND,* and *COOPER D. SCHMITT.*

PROBLEMS FOR SOLUTION.

ARITHMETIC.

128. **Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.**

At what time is the figure 7, on the face of a clock, midway between the hour and minute hands?

129. **Proposed by J. W. DAPPERT, Civil Engineer and Surveyor, Taylorville, Ill.**

"A Minion, agile, in stature small
Panting came to great Diana's Hall,
Bearing a marble globe upon his shoulders,
Measuring one inch in its diameters.
He rolled it to the northeast corner of the Hall,
Left touching the northern and eastern walls;
Then following came three demi-gods in white,
Each bearing a globe of lustrous metal bright;
One of iron, copper one, and one of silver;
And they placed them in the order given,
Touching each the other, and at the same time,
Touching each the side-walls, in a direct line,
The iron touching the marble, and its other side
Resting against the silver, in its glory and pride,—
All resting upon the oaken floor; and then
With heavy tread, and puff, and roar, Atlas came
Bearing a huge golden sphere, that filled the Hall,
Touching the four sides, floor and ceiling, and all
Radiant with beauty, resting against the silvery ball,
Making the globe's diameters in the rooms diagonal."

"Tell me, all ye who mathematics know:
What size the copper sphere, and oh!
How large the iron globe? How great
The golden globe, immaculate?
The silver sphere, how great? What size?
And if presented as a prize,
What value do you hold
Would be the sphere of gold?"

Found in an old volume entitled "Treatise on Surveying, by Abel Flint"—published in 1833, owned formerly by John E. Stockton, later by S. W. Baker, and now in my possession. Written in red ink upon a blank page.

*** Solutions of these problems should be sent to B. F. Finkel not later than May 10.

ALGEBRA.

117. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Rationalize $l^{\frac{1}{3}} + m^{\frac{1}{3}} + n^{\frac{1}{3}} + x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} = 0$.

118. Proposed by FREDERIC R. HONEY, Ph. B., Instructor at Trinity College, and Lecturer at Smith College, New Haven, Conn.

An army whose length is equal to a, moves forward. An officer is sent from the rear to the van, and is required to present himself at the rear again when the rear has reached the point where the van was when the army began to move. How far did the officer travel?

*** Solutions of these problems should be sent to J. M. Colaw not later than May 10.

GEOMETRY.

141. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

The equilateral triangle described on the hypotenuse of a right triangle is equivalent to the sum of the equilateral triangles described on the other two sides.

Prove without the aid of the famous Pythagorean proposition.

142. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Show that an infinite number of triangles can be inscribed in $x^2/a^2 + y^2/b^2 - 1 = 0$ whose sides touch $a^2x^2 + b^2y^2 = \frac{a^4b^4}{(a^2+b^2)^2}$.

143. Proposed by J. T. FAIRCHILD, A. M., Ph. D., Instructor in Mathematics, Crawfis College, Crawfis College, Ohio.

If the centers of three spheres do not lie in the same straight line, their surfaces cannot have more than two points in common. These points lie in a straight line perpendicular to the plane of centers and equal distances from this plane on opposite sides. [From *Phillips and Fisher's Geometry.*]

*** Solutions of these problems should be sent to B. F. Finkel not later than May 10.

CALCULUS.

108. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

The hypotenue of s plane right triangle increases uniformly at the rate of 1-12 of an inch a second. If the legs are as 2 to 3, at what rate is the area of the triangle increasing when the perpendicular from the right angle upon the hypotenuse is 12 inches?

109. Proposed by M. E. GRABER, Heidelberg University, Tiffin, Ohio.

Find the curve in which the product of the perpendicular drawn from two fixed points to any tangent is constant. (To be solved by using differential equations of the first order.)

110. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

Find the volume removed by boring an auger hole through a right circular cone, the radius of the auger being r, the radius of the cone R, and the altitude h, and the axis of the auger intersecting axis of the cone at right angles and at a distance c from the vertex.

*** Solutions of these problems should be sent to J. M. Colaw not later than May 10.

MECHANICS.

105. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $2a$, $2b$ be the diagonals of a rhombus, φ the angle the principal axis at the mid-point of a side makes with the diagonals. Prove $\tan 2\varphi = \frac{8}{3}\tan\beta$, when β is an angle of the rhombus. The principal moments of inertia about this mid-point of the side are $\frac{1}{24}m\{5a^2+5b^2 \pm_1 [25(a^2+b^2)^2-64a^2b^2]\}$.

106. Proposed by J. D. CRAIG, A. B., New Germantown, N. J.

The centers of the two wheels of a bicycle are three feet apart.

(1) If a rider wishes the rear wheel to trace a circle 14 feet in diameter, what must be the diameter of the circle traced by the front wheel?

(2) If the rider weighs 120 pounds, and his center of gravity is 4 feet from the ground, at what angle must he lean to make one revolution of the circle every 3 seconds?

*** Solutions of these problems should be sent to B. F. Finkel not later than May 10.

DIOPHANTINE ANALYSIS.

85. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Find a square proper fraction which if subtracted from unity will leave for remainder a square proper fraction.

86. Proposed by A. H. BELL, Hillsboro, Ill.

The edges of a rectangular parallelopiped are within one of the proportion $3 : 6 : 7$, and if they are $3x$, $6x \mp 1$, $7x$, then $(3x)^2+(6x \mp 1)^2+(7x)^2$ = the diagonal squared $= 94x^2 \mp 12x + 1 = \square$. To find four integral values of x in this equation.

*** Solutions of these problems should be sent to J. M. Colaw not later than May 10.

AVERAGE AND PROBABILITY.

93. Proposed by LON C. WALKER, Assistant Professor of Mathematics, Leland Stanford, Jr., University, Palo Alto, Cal.

In Problem 75, required the average area of the circle inscribed in the triangle.

94. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

Three points are taken at random on the surface of the sphere. Find the chance that the triangle thus formed is acute angled.

*** Solutions of these problems should be sent to B. F. Finkel not later than May 10.

MISCELLANEOUS.

88. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics in University of Tennessee, Knoxville, Tenn.

Sum to infinity the series $5\cos\theta + \frac{7\cos 3\theta}{3\,!} + \frac{9\cos 5\theta}{5\,!} \ldots\ldots$

89. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Find the earth's average density and mass, having given that the attraction of a ball of lead 2 feet in diameter, on a particle placed close to its surface, is less than the earth's attraction is the ratio 1 : 10250000, and that the density of lead is $11\frac{1}{2}$ times that of water.

90. Proposed by DR. E. D. ROE, Jr., Elmira, N. Y.

I shot my rifle at different ranges and found the following table of elevations e, for the vernier peep sight, for the given distances s:

s	e
0	21.0
100	24.5
200	28.5
300	33.5
400	40.0
500	48.5

The distances are measured in yards. How shall a table of elevations be constructed, giving the arguments e, for every five yards up to 500 yards? Do not give the whole table, but explain the method, and illustrate by giving a computation, carrying the result to three places of decimals. An actual problem.

*** Solutions of these problems should be sent to J. M. Colaw not later than May 10.

BOOKS AND PERIODICALS.

Synthetic Arithmetic. By Merritt S. Cook, C. E. 177 pages. Madison, Wis. Tracy, Gibbs & Co. 1899.

The following is the remarkable summary on the title page: "Containing many new principles and improved methods for computation of both simple and compound numbers, multiplication by methods of 'aliquot parts,' complements and partial products, division by substituted divisors, etc.; new method for *squaring* both simple and *compound* numbers, also mixed fractions; the 'basic' system of computing simple interest, by which no direct multiplication by the rate or time is required; also a symmetrical, comprehensive presentation of the Metrical System, aided by the use of algebraic symbols, together with new methods for conversion to and from the English system; finally brief articles on elements and problems connected with electro-motive and water power. Also various miscellaneous problems and new solutions both interesting and useful." We do not think

that the author's "improved methods" of solution will be very favorably received. In his attempt at brevity the writer of the book has "gone off on a tangent" until he is far away from anything that is practical for use in a good text-book. There are some good things in the book, but what is new is for the most part bad, at least for teaching purposes.

J. M. C.

Supplement to Advanced Arithmetic. By A. W. Plummer, M. D., Principal of Olivet Street School, Los Angeles, California. 86 pages. Price, 30 cents. Boston: D. C. Heath & Co. 1898.

This book was prepared to supplement the shortcomings of the Advanced Arithmetic in the California State Series. It contains a practical summary of advanced work, and will no doubt prove very satisfactory to teachers who need to supplement their work in various ways. J. M. C.

Primary Exercises in Arithmetic. Nos. 1, 2, 3 and 4. By H. J. Silver. New York, Cincinnati and Chicago: American Book Company. 1899.

These exercises are intended to supplement those of the text-books. They contain no problems, only the mechanical drill in the fundamental operations. The examples are already set down on the printed page, and the pupil needs only to record the answer in the blank space provided for the purpose. These exercises are well-suited for the little ones.

J. M. C.

Solution Book. By J. T. Fairchild, A. M., Ph. M., Instructor in Mathematics, Crawfis College, Crawfis College, Ohio. 255 pages. Published by the Author. 1898.

In this book solutions are given of a great number of problems that usually give teachers trouble. It covers less ground than some other books of its class, and is more particularly intended to give aid to the common school teacher. The problems have been taken from text-books, mathematical journals, and county examination tests. Several problems and solutions are reproduced from the *Monthly*. Many teachers will want to add this book to their collection. J. M. C.

Advanced Arithmetic. By William W. Speer, District Superintendent of Schools, Chicago. 261 pages. Price 60 cents. Boston: Ginn & Co. 1899.

In the series of which this is the Advanced book experiences in "relating" are at the basis of the treatment. Great prominence is given to the idea of *relative magnitude*, and hence "ratios" are kept constantly in view. The book has many excellent features, but in the attempt to make simple ratios the key to the solution of *all* problems the "ratio" method has been overworked. J. M. C.

Advanced Arithmetic. By E. McN. Carr. 373 pages. Prices, 45 cents. Richmond, Va.: B. F. Johnson Publishing Company. 1899.

Primary Arithmetic. By the same Author. 245 pages. Price, 25 cents.

The treatment of arithmetic as given in these books does not break from the "old methods." The arrangement is topical, and the plan of presentation proceeds upon the theory that principles should be clearly stated, and then illustrated by a reasonable number of appropriate examples and problems. The Primary book would have been better if classification had been made subordinate to gradation, at least in the earlier pages.

J. M. C.

Essentials of Arithmetic. By Gordon A. Southworth, Superintendent of Schools, Somerville, Mass. Book I, for Lower Grades, 186 pages, 40 cents; Book II, for Upper Grades, 311 pages, 60 cents. Boston: Thos. R. Shewell & Company.

These books have been received with remarkable favor. In the selection and arrangement of topics the author has shown a vigorous independence while his methods of presenting the material satisfactorily meets the requirements of the present time. The books are strong in oral work, up to date in business practice, and copious and varied in the supply of practical work and problems. J. M. C.

The Public School Mental Arithmetic. By J. A. McLellan, A. M., LL. D., and A. F. Ames, A. B. 138 pages. Price, 25 cents. New York: The Macmillan Company. 1899.

This book completes the series and completes the *method* of the text-books by the same authors based on the "Psychology of Number." The treatment is in strict line with the idea of number as *measurement*, and perhaps *undue* prominence is given to the *measuring* idea in some of the lessons. The exercises are varied and well graded. J. M. C.

American Elementary Arithmetic. By M. A. Bailey, A. M., Professor of Mathematics in the Kansas State Normal School, Emporia, Kansas. 208 pages. Price 35 cents. New York: American Book Company. 1898.

The pictures in this book are exceptionally good. The grading is not always well done—too much being required of the child in some places. The separate treatment of the fundamental operations is as unhappy as the combined treatment of the Grube method. The book has many valuable suggestions. J. M. C.

Plane Trigonometry and Tables. By Daniel A. Murray, B. A., Ph. D., Instructor in Mathematics in Cornell University, and formerly Scholar and Fellow at Johns Hopkins University. Crown 8vo., 219 + 95 pages, with a Protractor. Cloth, $1.25. New York, London and Bombay: Longmans, Green and Co. 1899.

The book deals with the subjects considered in the ordinary course in plane trigonometry, in colleges and secondary schools. Careful consideration has been given to the early difficulties and possible future needs of the beginner. In the practical applications, marked attention has been given to the graphical method of solution, as well as to the method of computation. The historical notes throughout the work are a valuable feature. The book is accurately written, and has many points of excellence that justify its appearance. J. M. C.

The following periodicals have been received: *The American Journal of Mathematics*, January, 1900; *The Educational Times*, March 1, 1900; *Journal de Mathématiques Élémentaires*, 1er Mars, 1900; *L'Intermédiaire des Mathématiciens*, Décembre 1899; *The Kansas University Quarterly*, October, 1899; *The Mathematical Gazette*, February, 1900; *Notes and Queries*, February, 1900; *Journal of Education*, January-March, 1900. J. M. C.

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A POPULAR ACCOUNT OF SOME NEW FIELDS OF THOUGHT IN MATHEMATICS.

By DR. G. A. MILLER.

Read at the regular Winter Term Meeting of the Alpha Chapter of Sigma Xi, Cornell University.

At the beginning of the nineteenth century elementary arithmetic was a Freshman subject in our best colleges. In 1802 the standard of admission to Harvard College was raised so as to include a knowledge of arithmetic to the 'Rule of Three.' A boy could enter the oldest college in America prior to 1803 without a knowledge of a multiplication table.* From that time on the entrance requirements in mathematics were rapidly increased, but it was not until after the founding of Johns Hopkins University that the spirit of mathematical investigation took deep root in this country.

The lectures of Sylvester and Cayley at Johns Hopkins University, the founding of the *American Journal of Mathematics* and the young men who received their training abroad coöperated to spread the spirit of mathematical investigation throughout our land. This has led to the formation of the American Mathematical Society eight years ago as well as to the starting of a new research journal, *The Transactions of the American Mathematical Society*, at the beginning of this year. While these were some of the results of mathematical activity, they, in a still stronger sense, tend to augment this activity.

In Europe such men as Descartes, Newton, Leibniz, Lagrange and Euler

*Cajori, *The History and Teaching of Mathematics*, 1890, page 60.

laid the foundation for the development of mathematics in many directions. These men, as well as a few of the most prominent names in the early part of the nineteenth century, were not specialists in mathematics. They were familiar with all the fields of mathematical activity in their day and some of them were well known for their contributions in other fields of knowledge. The last three-quarters of a century and especially the last two or three decades have witnessed a marvelous change in the mathematical activity of Europe. Mathematical periodicals have sprung up on all sides. A number of mathematical societies have been organized and many of the leading mathematicians have confined their investigations to comparatively small fields of mathematics.

The rapid increase of the mathematical literature created an imperative need of bibliographical reviews. This need was met in part by the establishment at Berlin, in 1869, of a year-book devoted exclusively to the reviews of mathematical articles, *Jahrbuch über die Fortschritte der Mathematik*. The 28th volume of this work reached our library a short time ago. It contains over 900 pages, and gives a review of over 2000 memoirs and books. With a view towards further increasing the facilities to keep in touch with the growing literature, the Amsterdam Mathematical Society commenced the publication of a semi-annual review, *Revue Semestrielle*, in 1893. In the last number of this, 236 periodicals are quoted, each of which contains, at times, mathematical articles that are of sufficient merit to be noted. Each of the four countries, France, Germany, Italy, and America publishes over thirty such periodicals.

One of the characterestic features of our times is the prominence of the spirit of coöperation. The mathematical periodicals and the mathematical societies are evidences of this spirit. In quite recent years international mathematical congresses have given further expression of the wide-spread desire to coöperate with even the most remote workers in the same fields. The first of these congresses was held in Zurich in 1897, and the second is to be held during the coming summer in connection with the Paris Exposition. The same spirit led in 1894 to the starting of a periodical, *L'intérmediaire des Mathématiciens*, which is devoted exclusively to the publishing of queries and answers in regard to different mathematical subjects.

This desire for extensive coöperation is tending towards unifying mathematics and towards laying especial stress on those subjects which have the widest application in the different mathematical disciplines. This explains why the theory of functions of a complex variable and the theory of groups are occupying such prominent places in recent mathematical thought.*

Before entering upon a description of some of the fields included in these subjects and the interesting problems which they present, it may be well to state explicitly that our remarks on mathematics will have very little reference to its application to other sciences. To the pure mathematician a result that has extensive application in mathematics is just as important and useful as one which applies to the other sciences. Mathematics is a science which deserves to be de-

*Cf. Klein, *Chicago Mathematical Papers*, 1893, page 134.